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The role of infinity (∞) in mathematics and allied disciplines

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Abstract:

This paper aims to analyse the idea of mathematical infinity (∞), its concept, definition, and applications in mathematics and allied disciplines, using published scientific journals. An analysis of the applications and interpretations of infinity across multiple disciplines will allow us a better understanding of the concept. The analysed literature shows that mathematical infinity is used in numerous fields of science, such as mathematics, economics, computer science, cosmology, physics, and many others. Emphasis is given to the theoretical understanding and uses of the term infinity, which are discussed.

Keywords: Infinity, mathematical infinity, cardinality, ZFC, aleph-null, continuum hypothesis

Introduction:

What is the idea of infinity (∞) about? Is infinity a number? Why does it appear in situations where numbers appear? Limits, integrals, sums, products, all these mathematical concepts are associated with ∞. We even know that some infinities are bigger than other infinities. But does that mean that $\infty + 1$ is actually bigger than ∞. Infinity is supposed to be the biggest number. How can something be even bigger? What would happen if we subtract ∞ from ∞? How do we even measure an infinite object in the first place? This paper is an attempt to answer these questions. In [1], (Allis & Koetsier, 1991, p. 187) discussed infinity beautifully, how a super task consisting of infinitely many steps can be completed in finite time. Zeno's dichotomy paradox is one of the many mathematical paradoxes which puzzled mathematicians and philosophers for centuries. Zeno claimed in his paradox that it is impossible to travel from one location to another. He points it this way. Suppose the speed of a person is 1 km/hr and the total distance to be covered is 1 km. Now the intuitive logic says the time taken to cover this distance is 1 hour. Zeno claims that half the distance is covered in $\frac{1}{2}$ an hour, half of the remaining distance is covered in $\frac{1}{4}$ an hour, half of the remaining distance is covered in $\frac{1}{8}$ hour, and so the process continues. So, if finite times are added infinitely many times, the total time taken to complete the journey will be infinity. In [3], (Angel, 2001, p. 347) discussed how the distance can be covered in 1 hr by considering only finite steps. In [4], (Bell, 2008) discussed infinitesimal calculus. It explores how a function can have a value where the input is infinitesimally close but does not coincide with a point where the function is undefined. In [5], (Blanché, 1964), infinity in metaphysics was discussed. Several mathematicians cited in the reference have discussed infinity in their own way, which results in the advancement of mathematics and science for all time. The most natural way to describe ∞ is to define a way to count objects, then use it to count collections that have infinitely many, like the natural numbers, integers, rational numbers, and real numbers, and that is exactly what sets can give us.

Main Theory and Analysis:

Sets are mathematical objects that contain other objects. {} is a set, specifically one that contains nothing. It is empty. We can then define zero as the number of elements in this set. More technically, we prefer to use the term cardinality. Similarly, we can define one as the cardinality of the set {*}, two as the cardinality of the set {*, *} and so on. But what if we put all the natural numbers into a single set? There is nothing about the definition of a set that prevents never-ending sets like {1, 2, 3,…}. This gives us a nice definition for infinity. If regular numbers are the cardinalities of sets with corresponding numbers of elements, then infinity is the cardinality of a set with a never-ending quantity of objects. We know there are many different finite cardinalities. But then, are there many different infinite cardinalities, or is there just one? To find out if this is the case, we need a way of telling whether two sets have the same cardinality, and then figure out if it is possible to have two infinite sets with different cardinalities. If two sets have the same finite cardinality, it is possible to pair each element in the first set with its own personal and unique element in the second set. It is also possible to do this the other way around. If the sets have different cardinalities, we cannot pair things up uniquely, resulting in a complicated situation. This is antithetical to many mathematicians' desires to avoid convoluted social situations, and this is contrary to the definition of sets with equal cardinalities. Nothing stops us from trying this property out on infinite sets. Even though there are infinite odd and infinite even numbers, we can define a one-to-one correspondence between them. It is pretty straightforward to prove that every element of the two sets is uniquely paired up, so they must have equal cardinality, meaning there are just as many even numbers as odd numbers, which makes sense. We can do the same thing between the even numbers and all of the integers. The conversion process is simply to multiply or divide the original number by two. Every even number is paired up with a unique integer, and every integer is paired up with the unique even number, and this is why infinity is so confusing. People often say that there are as many even numbers as there are integers. We use this language because it was the clear language to use when we were talking about finite sets, even though it does not necessarily make intuitive sense in the infinite case, using language like this. None of these terms makes consistent sense, and they are not supposed to. We continue to use the language of the simpler definition when applying it to new cases. But when we teach people about infinity and use these phrases without noting this, we create the false impression that there is supposed to be some intuitive notion of size that works on infinities.

The size of an infinite set is an abstract concept completely separated from its usage as a term in English. That is why we are going to continue to refer to this property. We are talking about cardinality rather than size in the hope that it helps emphasise that this concept does not really work the way the nomenclature of size suggests. So far, it is looking like it might be the case that there is only one infinite cardinality. If infinity is already bigger than all the other numbers, how can there be something even bigger? To confirm this, let's keep trying to make an infinite set that has a different cardinality than the set of all numbers. To start, it turns out that the rational numbers have the same cardinality as the integers. First, we consider only positive rational numbers. We can order them by starting at $\frac{1}{1}$, then going to $\frac{2}{1}$ and $\frac{1}{2}$, then going to $\frac{3}{1}, \frac{2}{2}$ and $\frac{1}{3}$ followed by $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}$ and $\frac{1}{4}$ and so on. This process is a way of pairing up any positive rational number with a natural number. We can extend this procedure to pair up zero and the negative rational numbers by simply mapping zero to zero and the negative integers to their corresponding negative rational numbers. Because we can convert between rational numbers and integers, there are as many rational numbers as there are integers. Our intuition of size tells us that making all the combinations of elements in a particular set should result in a much bigger set, but since we are unconstrained by intuitive notions of size, we find that it is actually possible to pair up an infinite set of elements with an infinite set of pairs of those elements. If making this bigger set resulted in the same infinite cardinality, we would always end up with the same infinite cardinality.

So, let's try one more thing. Let's try making a single sequence of all the real numbers. Our first idea might just be to try writing out all the real numbers in order, but then, which number comes after zero? Is it 0.1? But then we have already skipped over a bunch of other numbers. No matter how small we pick the second real number to be, there is always something in between. So, we will always skip over something. So, trying to write everything in order will not work. We also cannot use the trick we use to sort out all the rational numbers, as we would skip over irrational numbers like π. It does not seem to be possible to make a list of all the real numbers such that every real number gets its own integer-numbered spot on the list. It is impossible to use a finite amount of information to accurately represent an infinite amount of information, so we cannot pair up each real number with its own integer. To make a more formal argument, we can construct a proof by contradiction where we prove that the concept of a complete list of all real numbers contradicts itself. This proof is called Cantor's diagonalisation argument, and its generalised version will be helpful to us later. The fact that we cannot make a list of all the real numbers finally tells us that there is actually more than one infinite cardinality. Mathematicians invoke the use of Hebrew letters, specifically the cardinality of the integers, which is also the cardinality of the naturals and the rational numbers, is denoted by $\aleph_0$ (aleph-null), while the cardinality of the reals is denoted by c. We would like to point out that not only are these cardinalities different, but c is in some sense bigger than $\aleph_0$. A set has greater cardinality than another one if we cannot make a unique pairing between its elements. Now we have seen that not only are there many different infinite cardinalities, but some of them are actually bigger than others. This system of cardinalities is called the cardinal numbers. This name suggests that we can do arithmetic to add and multiply cardinal numbers, which is exactly what we are about to do, just like our definition of cardinality. We want our definition of arithmetic to produce the results we expect when we use it on finite numbers while keeping them general enough to work on infinite numbers as well. We can define what it means to add cardinalities by defining something to do on the sets that those cardinalities describe. Using this definition, we get an operation that behaves exactly like conventional arithmetic addition for finite numbers, but we can apply it to infinite numbers as well. For example, we can find the value of $\aleph_0 + 1$ by finding the cardinality of the natural numbers, with zero included in the result afterwards, which is just the non-negative integers, which also have cardinality $\aleph_0$. So $\aleph_0 + 1$ is $\aleph_0$. Similarly, we find that $\aleph_0 + \aleph_0$ is still $\aleph_0$ because combining the even and odd numbers gives us the natural numbers. Finally, $\aleph_0 + c$ is like combining the rationals with the irrationals to get the real numbers. So, we get $\aleph_0 + c = c$. Now we can define subtraction as the operation that undoes addition. But this is a bit tricky. As we have seen $\aleph_0 + 1$ is $\aleph_0$ but $\aleph_0 + \aleph_0$ is also $\aleph_0$. That means $\aleph_0 - \aleph_0$ could be 1, or it could be $\aleph_0$, or it could be any other number. Multiplication is defined as the cardinality of a set containing all pairings of elements from the first and second sets. So, two times three is six. Interestingly, $\aleph_0$ times itself is still just $\aleph_0$. Similar to what we did with the rational numbers, there are as many pairs of natural numbers as there are natural numbers themselves. c times itself is also c. Every pair of real numbers can be assigned its own unique real number and vice versa. So, the two sets must have equal cardinalities. Finally, raising 2 to the power of a cardinal number has an interesting interpretation on its corresponding set. If we take a set with two elements, we can recombine those elements into four different sets, while three elements can be recombined into eight different sets. In both cases, we have ended up with a corresponding power of two. So, raising 2 to the power of a cardinal number gives us how many new sets we can make only using elements of the original set. Previous operations seem to leave transfinite numbers intact, resisting any attempt to make bigger ones. This operation fixes that 2 to the power of $\aleph_0$ is now finally also c.

Now, a reasonable question is whether or not there is a cardinal number between $\aleph_0$and c. The answer is given by the continuum hypothesis, and it is no. While we should be able to prove it true or false. This question is not just hard to answer; it is literally impossible to fully appreciate this. Back in the early 1900s, mathematics was going through a bit of a crisis. At that time, different areas of mathematics were all based on different assumptions. Since they all talked about different kinds of objects, many proofs relied on intuition for some steps rather than formal arguments. Some mathematicians began trying to build a unified system of starting assumptions called axioms, which could be used to rigorously describe all other fields. But these attempts failed for one reason or another. Usually, the axioms allowed us to prove a statement that would be considered obviously false with the existing systems of mathematics, rendering the new system useless. Mathematician Gottlob Frege was just wrapping up the second of two volumes on a system of Mathematics, which he defined using axioms he felt were intuitive and obvious. As this book was just about to become available to the public, another mathematician informed him that one of his axioms could, in fact, be used to construct a famous paradox published two years earlier. This implied that truth and falsehood as defined in the system were actually the same thing. Technically, there were no contradictions in the system because it is always valid to say that a statement is true or false. It is just due to this paradox that everything is true and false at the same time. So, the concepts lose their meaning. Into this unfortunate situation entered mathematician David Hilbert and his program. Hilbert's program was his personal vision for what a unified system of mathematics should include. Among others, it included goals such as proving that the system we come up with can be trusted because it has no contradictions. We should prove that anything true can be proven with the system. It should come with an automated way to check the truth of simple mathematical statements, and these would all be very nice things to have. It would clear up all the problems they had at the time. But unfortunately, several seminal proofs completely obliterated any hope of accomplishing Hilbert's program. Mathematician Alfred Tarski showed that we cannot make a formula for telling the truth or falsity of a statement. If we could, nothing is stopping us from using that formula in a new statement, specifically, which the original formula by definition cannot accurately answer. Mathematician Kurt Gödel says that if a system can do even basic arithmetic, it can prove everything, and he proved that one of those unprovable statements is whether or not the system is trustworthy. We would have to come up with a more powerful system to answer that question. But then we don't know if the more powerful system might have its own contradiction. Once everyone got over that load of disappointment, they eventually settled on a good enough system of mathematics called ZFC. It might contain a lurking contradiction. But so far, no one has found one. So, it is probably the best. We just accept that everything in Mathematics comes with a little 99.99% likely to be true. This is better than most everything else, even though we cannot prove there are no contradictions. To be specific, the ZFC system is not strong enough to prove or disprove the Continuum hypothesis. If one tries to make a stronger system, one could always design it so that the Continuum hypothesis is true, or one could design it to make it false. Mathematicians have shown a few statements to be equivalent to the truth or falsity of the Continuum hypothesis. These results are relatively inconsequential, and we have not found any practical use for them. No Mathematician has ever made an effort to officially extend ZFC to accept or reject the Continuum hypothesis.

What Infinity means in $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. These kinds of equations mean we should do the operation forever until we get a single answer. But this definition does not really work. The definition of the sum requires adding together a bunch of numbers until we get an answer. But if we try to add an infinite quantity of numbers together, we will keep going on forever and never actually arrive at an answer. What we need is an extended definition of these operations that will let us make sense of infinity in this context. The way we do this is with a calculus tool called the limit, which lets us reason about infinity by only using finite numbers. When a function has a particular limit at a particular point, it says that the closer we get to that point, the closer the function gets to that limit. It doesn't matter if the function is not even defined at that exact point. The limit will tell us what probably exists there, given the value that is approached as we get closer. We can use this tool to analyse points on functions that we can get closer to but not actually arrive at. For example, $\frac{x}{x}$ is one everywhere except at zero, where it is undefined. But if we take the limit $\lim_{x \to 0} \frac{x}{x}$, we very conveniently get the value of one. Using this tool, we can plug infinity into all kinds of different operations without having to make definitions for what it means on a case-by-case basis.

Applications of Infinity Across Disciplines:

The concept or the idea of infinity (∞) is very technical in mathematics and many allied disciplines. The idea of infinity finds its use in mathematics, physics, cosmology, computer science, economics and many other disciplines. Although it is abstract, which can be observed but cannot be grasped fully, it helps in the formulation of a theoretical understanding. This concept can be used in understanding objects or situations having unbounded domains.

In mathematics, infinity finds its best use. The idea of infinity, although it is something that is limitless, is used in a tool called the limit. Limit of a function, limit of a sequence, limit (summation) of a series. A function may have a definite value if the input increases and increases endlessly. Similarly, a function may have a fixed value if the input approaches a value, although the function is undefined there. And it is this limit which is the foundation of differentiation and integration, forming an important branch of mathematics called calculus, and in advance form, it is mathematical analysis. Infinity plays a fascinating role in sets regarding their cardinalities. Two sets where one contains the other may have the same cardinality. The set of real numbers has a bigger cardinality than the set of natural numbers, the set of integers, or the set of integers or rational numbers. It is still undecidable whether there is a cardinality between the cardinalities of integers and real numbers.

In physics, we deal with nature, space, time and physical bodies governed by the laws of nature. Infinity plays a big role in dealing with those phenomena of nature governed by the laws of nature. Time is sometimes continuous and uncountable, sometimes infinitely small. Space is thought to be infinite. In generalising the ideas of our space and time, infinity plays a big role. Scientists are always curious to know about the origin of our universe, which many believe is the Big Bang. And here comes the concept of singularity. Singularity or singularities? Singularity is of infinite mass, infinite energy? One or infinitely many universes, each of infinite size and infinite energy? Scientists investigate these questions with the idea of infinity. In quantum field theory, infinitesimally small quantities are studied.

In computer science, the role of infinity is mostly abstract. In computer algorithms, if the size of the algorithms is infinitely large, computer scientists usually make use of a concept called asymptotic to understand the computational efficiency. Computer scientists make use of infinity to understand data structures, infinite loops, and finite automata in an infinite input set. Computer scientists also make use of infinity to understand the solution of the problems that involve infinitely many steps and those problems that are undecidable.

In economics, optimisation problems involving a long-term decision-making environment are solved by using the notion of infinity. In decision-making problems with indefinite future and game theory problems consisting of infinitely repeated interactions, economists use the concept of infinity in solving them.

Conclusion:

In this paper, we have discussed the concept of infinity, what it is all about, how it is understood and how it is used as a tool in mathematics and many other allied disciplines. Although it is unreachable and uncatchable, it can be observed. It helps in solving many problems in reality, and it is being used in understanding the nature and, most importantly, our own existence.

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